MATH 8610 (SPRING 2019) MIDTERM EXAM (TAKE-HOME)

Assigned 03/17/19, due 03/26/19 in class (Tuesday 8am).

1. Please mark the time you used (no pressure, just for information purpose).
2. You may refer to your lecture notes or homework now, but please get ready for similar problems on the final exam and comp math prelim (closed-book).
3. **NO discussion allowed with any other students. NO online resources.**
4. **Q3** 15 points; all others 10 points each.

1. **[Q1]** Let $f(x_1, x_2) = x_1^2 \ln x_2$ where $x_2 > 0$. Find the relative condition number of f. If $x_1 \approx 1$, for what values of x_2 is this evaluation ill-conditioned?

2. **[Q2]** Consider the Lyapunov matrix equation $AX + XA^T = Q$, where $A \in \mathbb{R}^{n\times n}$ is nonsymmetric, and $Q \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. The solution $X \in \mathbb{R}^{n\times n}$ must be real symmetric. Let $\widetilde{X}$ be a computed solution, which is the true solution of the equation with a slighted perturbed $A + \Delta A$ and the same Q. Show that the relative backward error $\frac{\|\Delta A\|}{\|A\|} \geq \frac{\|Q-(A\widetilde{X}+\widetilde{X}A^T)\|}{2\|A\|\|\widetilde{X}\|}$, and find a particular ΔA (no need to achieve the lower bound on the relative backward error).

3. **[Q3]** Assume that $U, V \in \mathbb{R}^{n\times p}$ $(p < n)$ have full rank, and $V^T U$ is nonsingular.
(a) Show that $P = I - U(V^T U)^{-1}V^T$ is a projector, and find range(P) and null(P).
(b) What can we say about P and $\|P\|_2$, if range(U) = range(V)?
(c) Instead, let $P = I - 2U(V^T U)^{-1}V^T$. Find the eigenvalues of P (please specify their multiplicities) and the expression of P^{-1}.

4. **[Q4]** Let $x \in \mathbb{R}^n$, and consider the vector $z = \begin{bmatrix} 0_{n-1} \\ 3\|x\|_2 \\ -4x \end{bmatrix} \in \mathbb{R}^{2n}$. Find the vector v that defines the Householder reflector $H = I - 2\frac{vv^T}{v^T v}$ such that Hz is a multiple of e_1. For $y = \begin{bmatrix} 0_n \\ x \end{bmatrix} \in \mathbb{R}^{2n}$, give a simplified expression of Hy.

5. **[Q5]** Given a 6×4 matrix A with all nonzero entries, illustrate the procedure of Golub-Kahan bidiagonalization, and explain how to compute all singular values of A.

6. **[Q6]** For $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$ of full column rank, show that $\|A^\dagger\|_2 = \frac{1}{\sigma_n(A)}$ (As usual, assume that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$).
(b) Let $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where $A_1 \in \mathbb{R}^{n\times n}$ is nonsingular. Show that $\sigma_n(A) \geq \sigma_n(A_1)$, and $\|A^\dagger\|_2 \leq \|A_1^{-1}\|_2$ (Hint: explore the relation between $\frac{\|Ax\|_2}{\|x\|_2}$ and $\frac{\|A_1 x\|_2}{\|x\|_2}$).

7. **[Q7]** Define the numerical rank of $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$ as
$$\text{rank}(A, \epsilon) = \max\{k : \sigma_k \geq \epsilon\}.$$
For a given ϵ, if A has numerical rank k $(k < n)$, find a *numerically* full rank B satisfying $\inf_{\text{rank}(B,\epsilon)=n} \|A - B\|_F$ and show that $\|B - A\|_F \leq (n-k)^{\frac{1}{2}}\, \epsilon$.