MATH 8610 (SPRING 2018) FINAL EXAM

Assigned 05/03/2019 at 12pm, due 05/06 at 10am. Late submission not accepted.

1. **[Q1]** Let $A \in \mathbb{R}^{n\times n}$ be real symmetric, indefinite and nonsingular. Consider a *signed* Cholesky factorization $A = LDL^T$, where L is lower triangular, and D is a diagonal matrix with ± 1 diagonal elements. Consider a collection of such matrices, for which $\kappa_2(L) \le C_n$ for some moderate constant $C_n > 0$ (assume n is fixed).

   (a) Show that for these matrices, the Cholesky factor L satisfies $\|L\|_2 \le \sqrt{C_n\|A\|_2}$.

   (b) Suppose a signed Cholesky factorization applied to these matrices gives $\widehat{L}$, and diagonal $\widehat{D}$ with ± 1 entries, such that $A + \Delta A = \widehat{L}\widehat{D}\widehat{L}^T$, with $\kappa_2(\widehat{L}) \le C_n$, and $\frac{\|\Delta A\|_2}{\|\widehat{L}\|_2\|\widehat{L}^T\|_2} = \mathcal{O}(\epsilon_{mach})$. Show this algorithm is backward stable for such matrices.

   (Hint: left and right multiply $A = LDL^T$ by L^{-1} and L, respectively, note that D is orthogonal, and find an upper bound on $\|L^TL\|_2$; also need $\|L\|_2\|L^T\|_2 = \|L^TL\|_2$)

2. **[Q2]** Let $A \in \mathbb{R}^{m\times n}$ $(m \ge n)$ be of full rank n, with SVD $A = \sum_{j=1}^n \sigma_j u_j v_j^T$, with singular values $\sigma_1 \ge \sigma_2 \ge \dots \sigma_n > 0$. Choose and fix index k $(1 \le k < n)$, define $A_k = \sum_{j=1}^k \sigma_j u_j v_j^T + \sum_{j=k+1}^n \frac{\sigma_{k+1}}{2} u_j v_j^T$, and consider $S = \{B : B \in \mathbb{R}^{m\times n}, \sigma_j(B) = \frac{\sigma_{k+1}}{2}, k+1 \le j \le n\}$ (similarly, assuming that $\sigma_1(B) \ge \dots \ge \sigma_n(B)$). Show that

   $$\|A - A_k\|_2 = \inf_{B\in S} \|A - B\|_2.$$

   (Hint: Note that $\|Aw\| \le \|Bw\| + \|(A-B)w\|$, assume that there exists a minimizer $B \ne A_k$, and let w lie in a subspace spanned by certain right singular vectors)

3. **[Q3]** Consider the unshifted QR iteration applied to a real symmetric tridiagonal matrix H, described by $Q^{(k)}R^{(k)} = H^{(k-1)}$ and $H^{(k)} = R^{(k)}Q^{(k)}$, with $H^{(0)} = H$. Define $\underline{Q}^{(k)} = Q^{(1)} \cdots Q^{(k)}$ and $\underline{R}^{(k)} = R^{(k)} \cdots R^{(1)}$.

   (a) Is the arithmetic work of each QR iteration $\mathcal{O}(n)$, $\mathcal{O}(n^2)$, or $\mathcal{O}(n^3)$, and why?

   (b) With $H^k = \underline{Q}^{(k)}\underline{R}^{(k)}$, show that under certain mild assumptions, the first and the last column of $\underline{Q}^{(k)}$ converge to the eigenvector of H associated with the largest and the smallest (modulus) eigenvalues, respectively.

   (c) Now consider the *shifted* QR iteration. Assume that the bottom-right 3×3 block of $H^{(k)}$ is $\begin{bmatrix} \times & \eta a & \\ \eta a & a+b & \delta \\ & \delta & b \end{bmatrix}$, with $|\delta|$ sufficiently small, $|a|$ not very small, and $\eta \ne 0$. Assume that the shift $\mu^{(k+1)} = b$ is used to transform $H^{(k)}$ to $H^{(k+1)}$. Give an upper bound on the $(n, n-1)$ entry of $H^{(k+1)}$ in modulus. What does this imply?

4. **[Q4]** Consider the Arnoldi relation $AU_k = U_{k+1}\underline{H}_k$, with $U_k^TU_k = I$, $\underline{H}_k \in \mathbb{R}^{(k+1)\times k}$. Let (μ, w) be an eigenpair of H_k (the top k rows of $\underline{H}_k$).

   (a) Show that (μ, U_kw) satisfies $AU_kw - \mu U_kw \perp \mathcal{K}_k(A, u_1)$, and $\|AU_kw - \mu U_kw\|_2 = |h_{k+1,k}w(k)|$, where $w(k)$ is the last element of w.

   (b) What happens if $\mathrm{col}(U_k)$ is an invariant subspace of A, i.e., $\mathrm{col}(AU_k) \subset \mathrm{col}(U_k)$? Under what condition(s) for u_1 would this scenario happen?

5. **[Q5]** Let $r_0 = b - Ax_0$ be the initial residual vector of the linear system $Ax = b$, and $r_k = r_0 - Az_k$ with $z_k \in \mathcal{K}_k(A, r_0)$.

   (a) Show that $p_k \perp A\mathcal{K}_k(A, r_0)$ for CG, and $r_k \perp A\mathcal{K}_k(A, r_0)$ for GMRES. As a result, show that $(r_j, p_k) = (r_k, p_k)$ for CG, and $(r_j, r_k) = (r_k, r_k)$ for GMRES $(1 \le j < k)$.

(b) Let $AU_k = U_{k+1}\underline{H}_k$ be the Lanczos/Arnoldi relation for solving $Ax = b$, where $u_1 = \frac{r_0}{\|r_0\|_2}$. Let the k-th iterate of CG or GMRES be $x_k = x_0 + U_k y_k$. Show that $H_k y_k = \|r_0\|_2 e_1$ for CG, whereas $\underline{H}_k^T \underline{H}_k y_k = \|r_0\|_2 \underline{H}_k^T e_1$ for GMRES.

(Hint: for GMRES, consider the normal equation for the linear least squares)

6. **[Q6]** Let $A \in \mathbb{R}^{n\times n}$ be real symmetric, indefinite and nonsingular. Consider using the MINRES algorithm to solve the linear system $Ax = b$ iteratively. Assume that the eigenvalues of A all lie in the intervals $[-a, -b] \cup [c, d]$, with $-a < -b < 0 < c < d$, and $a - b = d - c$ (two intervals are of the same length). Let $r_0 = b - Ax_0$ and $r_{2k} = b - Ax_{2k}$ be the initial and the $2k$-th residual of MINRES. Show that

$$\frac{\|r_{2k}\|_2}{\|r_0\|_2} \leq 2\left(\frac{\sqrt{ad} - \sqrt{bc}}{\sqrt{ad} + \sqrt{bc}}\right)^k .$$

What does this bound suggest if A has a large 2-condition number?

(Hint: let $q_2(t) = 2\frac{(t+b)(t-c)}{bc-ad} + 1$, $p_{2k}(t) = \frac{T_k(q_2(t))}{T_k(q_2(0))}$; show that $q_2(-a) = q_2(d) = -1$, $q_2(-b) = q_2(c) = 1$, and recall that MINRES is mathematically GMRES)