MATH 8600 (FALL 2018) HOMEWORK 7

Assigned 11/19/18, due 11/29/18 in class.

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1. **[Q1]** Suppose that the degree of accuracy of a quadrature rule with $n+1$ nodes is m, and all $w_k > 0$. Then for $f \in C^{m+1}([a,b])$, the error of the quadrature satisfies

$$\left| \int_a^b f(x)dx - Q(f) \right| \le \frac{m+3}{2^{m+1}(m+2)!} \max_{c\in[a,b]} \left|f^{(m+1)}(c)\right|(b-a)^{m+2}.$$

(Hint: consider the Taylor expansion of $f(x)$ at $\frac{a+b}{2}$, a polynomial of degree m plus a remainder of order $m+1$. Then $\int_a^b f(x)dx - Q(f)$ is the difference between the integral and the quadrature of the remainder, bounded by the sum of the two. Also, $\sum_{k=0}^n |w_k| = \sum_{k=0}^n w_k = b-a$. Here, we have a desirable additional factor 2^{m+1} compared to the result mentioned in class, due to the expansion at the midpoint.)

2. **[Q2*]** Assume that a quadrature rule has both positive and negative weights, denoted by w_k with $k \in S^+ \subset \{0,1,\dots,n\}$ and $k \in S^- = \{0,1,\dots,n\} \setminus S^+$, respectively. Suppose that the largest positive and negative (in absolute value) weights both go to infinity as $n \to \infty$. For any given $\delta > 0$ (small) and $M > 0$ (large), show that there exist n, and $f, g \in C[a,b]$ with $\|f-g\|_\infty \le \delta$, such that $|Q(f) - Q(g)| \ge M$. Comment on the importance of the positiveness of quadrature weights.

3. **[Q3]** (a) Use the 2nd order Taylor expansion of $f(x)$ at $\frac{a+b}{2}$ to show that the error of the midpoint rule is $\int_a^b f(x)dx - Q(f) = \frac{(b-a)^3}{24} f''(c)$ for some $c \in (a,b)$.
(b) Let $p_1(x)$ be the linear interpolant of $f(x)$ at $x_0 = a$ and $x_1 = b$. Use the error $f(x) - p_1(x)$ to show that the error of the trapezoidal rule is $-\frac{(b-a)^3}{12} f''(c)$.

4. **[Q4*]** (a) Prob 15.4(a) (Hint: use the error of Hermite interpolation)
(b) The order of precision of Simpson's rule is $m = 3$, because it has 3 nodes ($n = 2$), $x_1 = \frac{a+b}{2}$ is the midpoint, and x_0 and x_2 are symmetric with respect to x_1 (see notes). Let $p(x)$ be the Hermite interpolation of the integrand $f(x)$, interpolating $(a, f(a))$, $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$, $\left(\frac{a+b}{2}, f'\left(\frac{a+b}{2}\right)\right)$, and $(b, f(b))$. What is the relation between $\int_a^b p(x)dx$ and the quadrature $Q(p)$? Use the error formula for $f(x) - p(x)$ to find the error of the Simpson's rule and compare with the one on textbook p. 445.

5. **[Q5]** (a) Derive the Simpson's rule by hand (take 3 equispaced nodes on $[a,b]$, and evaluate $w_k = \int_a^b L_k(x)dx$). Then, use symbolic math software (such as Mathematica or www.wolframalpha.com) to verify the Boole's rule.
(b) Use the uploaded code `newtoncotes.m` to compute the Newton-Cotes quadrature of $\int_{-1}^1 \frac{dx}{5x^4+4x^3+3x^2+2x+1}$, using up to 21 equispaced nodes, and compare with the integral value $I(f) = 1.615636766490166$. Do Newton-Cotes rules give accurate result? What if we use these rules to approximate $\int_{-3}^{-1} \frac{dx}{5x^4+4x^3+3x^2+2x+1}$? Explain your observations. Should we try Newton-Cotes of higher order if the results are not satisfactory? (Refer to **[Q2]** and Lagrange interpolation based on equispaced nodes)

6. **[Q6]** (a) By hand and with MATLAB **as a calculator**, use the composite trapezoidal and Simpson's rules to approximate $\int_{-1.5}^{1.5} \frac{dx}{5x^4+4x^3+3x^2+2x+1}$, with $\ell = 6$ subintervals. Please show detailed expressions of both quantities and the values of each term.

   (b) Prob 15.6(a) (compare with the composite corrected rules given in class).

7. **[Q7]** (a) Write MATLAB code for the composite trapezoid (uncorrected & corrected), Simpson's, and Boole's rules to approximate the integral in **[Q5](b)**, $\int_0^1 \sqrt{x}dx = \frac{2}{3}$, and $\frac{1}{2\pi}\int_0^{2\pi} \sqrt{1-0.36\sin^2 x}dx = 0.9027799277721939$. Take $\ell = 32, 64, 128, 256$ and 512 equispaced subintervals, and find $|I(f) - Q(f)|$ for each ℓ and each quadrature. How much does the error decrease when ℓ is doubled for each quadrature, and why?

   (b) Perform the Romberg quadrature, making the Romberg table with $\ell = 512$ equispaced subintervals (no more subdivisions).

8. **[Q8*]** Derive the 3-point Gauss-Legendre quadrature for $\int_{-1}^{1} f(x)dx$, and the 2-point Gauss-Chebyshev quadrature for $\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx$. Please follow the order of precision of Gauss quadratures, and use the symmetry of $[-1, 1]$ and the weight function $w(x)$.

9. **[Q9]** (a) Consider the Clenshaw-Curtis quadrature. The **original** quadrature nodes $\{x_i\}_{i=0}^n$ on $[-1, 1]$ are the Chebyshev points $\cos\left(\frac{i\pi}{n}\right)$. For even n, we have

$$w_i = \begin{cases} \frac{1}{n^2-1} & i = 0 \text{ or } n \\ \frac{2}{n}\left[1 - \left\{\sum_{k=1}^{n/2-1} \frac{2}{4k^2-1}\cos\left(\frac{2ik\pi}{n}\right)\right\} - \frac{(-1)^i}{n^2-1}\right] & 1 \le i \le n-1 \end{cases}$$

   and for odd $n \ge 3$, the formulas are

$$w_i = \begin{cases} \frac{1}{n^2} & i = 0 \text{ or } n \\ \frac{2}{n}\left[1 - \left\{\sum_{k=1}^{(n-1)/2} \frac{2}{4k^2-1}\cos\left(\frac{2ik\pi}{n}\right)\right\}\right] & 1 \le i \le n-1 \end{cases}$$

   Give the 3-node and 4-node Clenshaw-Curtis quadrature rules for $\int_a^b f(x)dx$. Then use them to approximate $\int_{0.19}^{0.29} e^{-5x}\sin\frac{1}{x}\sin\frac{1}{\sin\frac{1}{x}}dx = 0.02115910278025609$.

   (b) Consider the Gauss quadrature with 4 ($n = 3$) nodes. Use the three-term recurrence formula of the Legendre polynomials $\phi_{n+1}(x) = \frac{2n+1}{n+1}x\phi_n(x) - \frac{n}{n+1}\phi_{n-1}(x)$ to derive the expression of $\phi_4(x)$, and use MATLAB's `roots` function to find the quadrature nodes numerically. Then, compute the weights $w_i = \frac{2(1-x_i^2)}{[(n+1)\phi_n(x_i)]^2}$ $(0 \le i \le n)$ on MATLAB. Use this Gauss quadrature to approximate the integral in part (a).

10. **[Q10*]** Prob 11.9 (derive a difficult 2-node weighted Gauss quadrature).

11. **[Q11]** (a) To construct a composite Gauss quadrature rule of order 20, how many nodes are needed on each subinterval? Use the uploaded `gausslg` to compute the nodes and weights of the corresponding (original) Gauss quadrature on $[-1, 1]$, and then write MATLAB code for this composite quadrature. By trial and error, find how many equispaced subintervals and total number of quadrature nodes are needed to approximate $\displaystyle\int_{0.1593}^{0.3182} e^{-5x}\sin\frac{1}{x}\sin\frac{1}{\sin\frac{1}{x}}dx = 0.02561655631847027$ to full precision.

(b) Run the uploaded `HW7_quadcomp` to compare MATLAB's `integral`, composite Simpson's rule, classical (eigenvalue-based) and fast Gauss quadrature, fast Clenshaw-Curtis quadrature (FFT-based), and the application of these rules on three subintervals ($f(x)$ changes mildly on the middle subinterval but rapidly elsewhere).

To get accurate timing, close all other applications, use a for loop repeating the code 100 times and take the average time for each quadrature rule.

Report the number of nodes and the average time for each quadrature algorithm, and draw some conclusions (including the possible reuse of quadrature nodes as n doubles for Clenshaw-Curtis, even though this is not explored here).