# 1 Basic Understanding

1. Answer the following

   (a) What points is the function $f(x) = \frac{x^2-4}{x+2}$ undefined at if any?

   (b) What does it mean algebraically for $F(x)$ to be the antiderivative of $f(x)$?

   (c) What is the definition of a critical point?

   (d) Write down the definition of a derivative of $f(x)$ at $x = x_0$.

   (e) Let $s(t), v(t)$ and $a(t)$ represent the position, velocity and acceleration of a car at time $t$. Describe their relationship using calculus.

   (f) Can the intermediate value theorem be applied to $h(t) = \frac{1}{x}$ over $(1, 4)$?

   (g) Let $v(t)$ be the velocity of a car at time $t$. Graphically, what represents the total distance traveled of the car? Furthermore, how would you go about graphing $s(t)$ (the position of the car at time $t$)

   (h) How are the tangent line at a point and the secant line related?

   (i) $\frac{1}{b-a}\int_a^b f(x)dx$ is referred to as what quantity of $f$?

   (j) Write down the fundamental theorem of calculus.

   (k) Draw a graph of a nondifferentiable function.

   (l) If $x = 4$ is a local max of $f(x)$, then what must be true about the derivative of $f$ at $x = 4$? What about the second derivative?

   (m) What is the slope of secant line that passes through $s(t)$ at $t = 0$ and $t = 2$?

2. True or False

   (a) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_b^c f(x)dx$

   (b) $k\int f(x)dx = \int kf(x)dx$

   (c) $\frac{d}{dx}f(g(x)) = f'(x)g'(x)$

   (d) $(f+g)'(x) = f'(x) + g'(x)$

   (e) $(a-b)\max f(x) \leq \int_b^a f(x)dx \leq (a-b)\min f(x)$

   (f) $\int f(x)g(x)dx = \int f(x)dx \int_a^b g(x)dx$

3. Evaluate the following without a calculator

   (a) If $p(t) = t^2 + ct$ and $p(4) = 2$, then what is $c$?

   (b) $\int_0^4 x^2 dx$

   (c) Find $f'(x)$ if $f(x) = e^{e^x}$

   (d) Evaluate $h(12)$ if $h(x) = kx^2 + 4x + k$ for some constant $k$.

   (e) $\frac{d}{dx}\int_4^{2x} \frac{12}{t^2}dt$

   (f) Find the antiderivative of $f(x) = x^4 + 3x^2 + 1$

   (g) Solve $x^2 - 6x + 5 = -4$.

   (h) $\int_0^{\frac{\pi}{2}} \sin(x) + \cos(x)dx$

(i) $\int_{-4}^{4} \frac{1}{2x} dx$

(j) Verify that $f$ is continuous at $x = 2$ if $f(x) = \begin{cases} x^2 & x < 2 \\ 4x - 4 & x \geq 2 \end{cases}$

4. Answer the following with **coherent** work and notation.

(a) Let $s(t), v(t)$, and $a(t)$ represent the position, velocity and acceleration of Rachel's serve at time $t$. She makes contact with the ball at 10 meters above ground and that $a(t) = -10$ (from gravity). If we assume that "they face" is 5 meters above ground, how long does it take for the ball to hit "they face" if she hits with an initial velocity of 20 meters per second? Here's a few subquestions to help you break it down.

- Draw a picture.
- There are three "initial conditions" given in the problem statement. Identify them all.
- What is the question asking you to find (specifically in terms of $s(t)$).
- If you know $a(t)$ (which you do, I gave it to you), how can you find $s(t)$?

After answering the above, solve the following questions using the same position function

- Use the position from the previous problem to find the highest point the ball reaches during the serve.
- Using the same position function for the previous two problems, verify that at some point, the ball is exactly 7 meters off the ground

(b) Let $v(t) = t^2 + 9t$ be the velocity of a car. Find a time $t_0$ such that the speed of the car at time $t_0$ is equal to its average speed over the interval $[0, 4]$.

(c) This problem and the next showcase the effectiveness of Riemann sums to approximate integrals. Let $f(x) = 6x + 4$. Find the area under the curve of $f(x)$ from 0 to 10 through 3 different methods.

- Directly compute it using a graph
- Use the fundamental theorem of calculus
- *Approximate* it using a midpoint Riemann sum with $n = 5$ subintervals.

(d) Notice that our "approximation" is exact in this case. This is not a coincidence. Argue that this is always true for linear functions by showing that the average value $f(x)$ occurs at its midpoint. We can use this fact to derive a formula for the sum of the first $n$ integers through the following.

- Show that the sum of the first $n$ integers is the same as taking the area under the curve of a piecewise function (draw it).
- Use the previous trick to argue that the sum is just the area under the curve of a particular linear function that passes through the midpoints of each interval.
- Find the area under the curve from 0 to $n$.

(e) COVID 19 has hit the aluminum business hard, and Pepsi is in a financial crisis. They need to cut back on costs of making a 12 oz can of Pepsi. They are introducing a new, perfectly cylindrical, can and have tasked their mathematicians with finding the right dimensions to minimize the cost of the can. Find the dimensions of the can that minimize the cost of a 12 oz can of Pepsi. You may assume that any object with volume $128\text{cm}^3$ can hold 12 oz of soda.

(Hint: a perfect cylinder is parameterized by a radius $r$ and a height $h$. The volume of a cylinder is $\pi r^2 h$ and its surface area is $2\pi r^2 + 2\pi rh$. Draw a picture and write out the model. It will help. )

(f) Cathy was pulled over by a cop claiming she had been speeding through a school zone. She is confirmed passing two cameras 2 miles apart in 5 minutes. What does the mean value theorem say about her speed? If the speed limit of the school zone was 35 miles per hour, did she speed?